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Gravitational radiation in Szekeres's quasi-spherical space–times†

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Abstract. A necessary and sufficient condition for flatness of the quasi-spherical space–times of Szekeres is given. Using this, the linear approximation of these space–times is considered and it is shown that the third derivatives of the quadrupole moments vanish for matter distribution contained within a comoving surface. Hence the Einstein formula gives zero energy loss due to gravitational radiation. This agrees with other investigations, which have shown that no radiation is present in these space–times.

1. Introduction

Although it is now widely accepted that gravitational waves do occur in the context of general relativity, one of the outstanding questions is whether freely falling matter radiates or not (Bonnor 1976, Ehlers *et al* 1976). Einstein's equations for pressureless matter or 'dust' are a good model for freely falling matter, but the known exact solutions of these equations have not yet given a definitive answer. The work of Cocke (1966), who reported the cylindrically symmetric solutions as radiative, has been questioned on physical grounds (Bonnor 1976). The other known dust solutions have, with one exception, high spatial symmetry, so radiation would not be expected. The exception is the Szekeres (1975a, b) class of exact solutions which do not have any Killing vectors (Bonnor *et al* 1977).

Recently (Bonnor 1976) it has been shown that a spherical portion of finite spatial volume of a solution of the 'quasi-spherical' class of Szekeres solutions can be smoothly matched to an exterior Schwarzschild metric. This metric is static and hence these spaces are non-radiative. A similar conclusion has been reached with different arguments by Berger *et al* (1977).

Although several approximation methods show that gravitational waves are produced for certain motions of matter, these do not apply to free fall (Bonnor 1963). Approximation methods designed for free fall have not yet been shown to be satisfactory. In particular, it is not known whether Einstein's linear approximation formula

$$dE/dt = \tfrac{1}{45}\dddot{D}^{\alpha\beta}\dddot{D}^{\alpha\beta}, \qquad (1.1)$$

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where $D^{\alpha\beta}$ are the quadrupole moments and E is the energy of radiation, is valid for free fall. In this paper we evaluate the formula (1.1) for the freely-falling matter in a quasi-spherical Szekeres space–time and show that it is consistent with the conclusion mentioned above—that there is no radiation.

In § 2 the quasi-spherical space–times of Szekeres are introduced. Section 3 contains the conditions under which the gravitational field is 'weak'. In § 4 we look for Minkowski coordinates for the flat space background of the linear approximation used and in § 5 we evaluate dE/dt given by (1.1). The paper ends with a brief conclusion.

2. Szekeres quasi-spherical space–times

Szekeres found two classes of solutions of Einstein's equations for dust with zero cosmological constant (later generalised to include perfect fluid sources by Szafron and Wainwright (1977); see also Szafron 1977). The equations are

$$R^{ik}-\tfrac{1}{2}g^{ik}R=-8\pi\rho u^i u^k, \tag{2.1}$$

where R^{ik} is the Ricci tensor, g^{ik} the metric tensor, $R=R^i i$ is the Riemann scalar, ρ is the density of the pressureless matter, and u^i is the unit four-velocity. Comoving coordinates are chosen, so $u^i=\delta^i_4$. The metric is of the form

$$ds^2=-\exp(2\alpha)\,dr^2-\exp(2\beta)(dy^2+dz^2)+dt^2. \tag{2.2}$$

For the class of solutions in which we are interested ($\partial\beta/\partial r\neq 0$), α and β are given by

$$\exp(\beta)=\frac{\phi(r,t)}{P(r,y,z)}; \qquad \exp(\alpha)=\frac{P}{w(r)}\frac{\partial}{\partial r}(\exp(\beta)), \tag{2.3}$$

with

$$P=a(r)(y^2+z^2)+2f(r)y+2g(r)z+c(r), \tag{2.4}$$

and the quasi-spherical subclass is further determined by the condition

$$ac-f^2-g^2=\tfrac{1}{4}, \tag{2.5}$$

a, c, f and g being arbitrary functions of r.

The dynamical properties of the system are governed by the equation for ϕ

$$\phi_4^2=w^2-1+S(r)/\phi; \tag{2.6}$$

throughout this paper suffices 1, 2, 3, 4 will denote differentiation with respect to r, y, z and t respectively. (2.6) is a *Friedmann* equation, and is the same as that obtained in spherically symmetric collapse of dust (Tolman 1934). Apart from the restriction

$$w>0, \tag{2.7}$$

the six functions a, f, g, c, w, S are arbitrary functions of their argument. A seventh function arises upon integration of (2.6). Of these only five are independent because there is a relation (2.5) between them and, moreover, one function of r can be given an assigned form by a coordinate transformation $r'=F(r)$.

The density is given by

$$8\pi\rho=\frac{(PS_1-3SP_1)}{\phi^2(P\phi_1-\phi P_1)}. \tag{2.8}$$

3. Conditions for a weak field

We shall state the flatness condition of a Szekeres quasi-spherical space–time, and then find weak field solutions as small departures from flatness.

By a detailed calculation of the Riemann tensor R_{ijkl} we have found

Theorem. *$S=0$ is a necessary and sufficient condition for a Szekeres quasi-spherical space–time to be flat.* If $S=0$

$$\phi = e(w^2-1)^{1/2}t + h(r), \tag{3.1}$$

where $e=\pm 1$, and h is an arbitrary function of integration.

From (3.1), which follows directly from (2.6), we see that we have further to restrict w as

$$w \geqslant 1. \tag{3.2}$$

We shall suppose that the weakness of the field is governed solely by the magnitude of S, and write

$$S(r) = \epsilon \tilde{S}(r), \tag{3.3}$$

where ϵ is a small parameter. $\tilde{S}(r)$ and the remaining arbitrary functions will not be assumed small.

The metric will be expressed as

$$g_{ik} = \eta_{ik} + \gamma_{ik}, \tag{3.4}$$

where η_{ik} is the metric for flat space–time in Szekeres coordinates, that is (2.2) with ϕ given by (3.1), and γ_{ik} is a small deviation from flatness of order ϵ.

With (3.3), (2.6) gives ϕ to order ϵ as

$$\phi(r,t) = \phi^0 + \epsilon\tilde{\phi}(r,t), \qquad \phi^0 = e(w^2-1)^{1/2}t + h, \tag{3.5}$$

where $\tilde{\phi}$ is determined by the differential equation

$$2e(w^2-1)^{1/2}\tilde{\phi}_4 = \tilde{S}/[e(w^2-1)^{1/2}t + h]. \tag{3.6}$$

The metric becomes, to first order

$$ds^2 = -\exp(2\alpha_0)\,dr^2 - \exp(2\beta_0)(dy^2+dz^2) + dt^2 - \epsilon[\Phi\,dr^2 + \Omega(dy^2+dz^2)], \tag{3.7}$$

where $\exp(\alpha_0)$ and $\exp(\beta_0)$ are as in (2.3) with $\phi=\phi^0$ and

$$\Phi = 2\frac{P^2}{W^2}\left(\frac{\tilde{\phi}}{P}\right)_1\left(\frac{\phi^0}{P}\right)_1; \qquad \Omega = 2\frac{\tilde{\phi}\phi^0}{P^2}, \tag{3.8}$$

whereas the density becomes

$$8\pi\rho = \epsilon\frac{(P\tilde{S}_1 - 3\tilde{S}P_1)}{(\phi^0)^2(P\phi^0_1 - \phi^0 P_1)}. \tag{3.9}$$

We have been using units in which $G=1$ and $c=1$ (see (2.1)). If we use units in which $G\neq 1$ and $c=1$ we should write $8\pi G\rho$ in (2.1) which, by (3.9), justifies us in identifying the parameter ϵ with the gravitational constant G.

To evaluate the quadrupole moments of the linear approximation we shall need Cartesian coordinates for the flat space background. The equations of transformation will be considered in the next section.

4. Equations of transformation to Minkowski coordinates

Consider a flat space–time. Let $X^k : X^1 = X$, $X^2 = Y$, $X^3 = Z$, $X^4 = T$ denote a set of Minkowski coordinates, and $r^a : r^1 = r$, $r^2 = y$, $r^3 = z$, $r^4 = t$ a set of Szekeres coordinates as in (2.2). In the following, a primed symbol for a quantity means that it is to be calculated in X^k; unprimed quantities are to be calculated in r^a.

If we put

$$\partial X^k/\partial r^a = P^k{}_a, \tag{4.1}$$

the equations of transformation for the Christoffel symbols,

$$\frac{\partial^2 X^k}{\partial r^a \partial r^b} = \left\{ {c \atop ab} \right\} \frac{\partial X^k}{\partial r^c} - \left\{ {k \atop lm} \right\}' \frac{\partial X^l}{\partial r^a} \frac{\partial X^m}{\partial r^b}, \tag{4.2}$$

become

$$\frac{\partial P^k_a}{\partial r^b} = \left\{ {c \atop ab} \right\} P^k{}_c, \tag{4.3}$$

because $\{\ \}' = 0$. Hence the problem reduces to the determination of the 20 functions X^k, $P^k{}_a$ satisfying the linear system of differential equations (4.1) and (4.3), and also the 10 finite equations

$$\eta_{ab} - \eta'_{ik} P^i{}_a P^k{}_b = 0, \tag{4.4}$$

which are the equations of transformation of the metric tensor η_{ab}. The integrability conditions of (4.1) are satisfied identically because of (4.3), and the conditions of integrability of (4.3) are

$$R_{abcd} - R'_{ijkl} P^i{}_a P^j{}_b P^k{}_c P^l{}_d = 0, \tag{4.5}$$

where R_{abcd} is the Riemann tensor, which are also satisfied.

All the Christoffel symbols $\{ {c \atop 44} \}$ vanish, so from (4.3) we have

$$\partial P^k{}_4/\partial t = 0. \tag{4.6}$$

Hence from (4.1) we see that the transformation must be linear in t:

$$X^k(r, y, z, t) = \bar{X}^k(r, y, z)t + \bar{\bar{X}}^k(r, y, z). \tag{4.7}$$

(4.7) is all the information we shall need to prove the main result of the paper in the next section.

We shall need the transformation from the coordinates r, y, z, T to Minkowski coordinates. Putting $k = 4$ in (4.7) we obtain

$$t = \tilde{T}(r, y, z)T + \tilde{\tilde{T}}(r, y, z), \tag{4.8}$$

with

$$\tilde{T} = 1/\bar{T}, \qquad \tilde{\tilde{T}} = -\bar{\bar{T}}/\bar{T}. \tag{4.9}$$

This, with (4.7), gives the transformation

$$X^\alpha(r, y, z, T) = \tilde{X}^\alpha(r, y, z)T + \tilde{\tilde{X}}^\alpha(r, y, z), \qquad \alpha = 1, 2, 3, \qquad T = T, \tag{4.10}$$

with

$$\tilde{X}^\alpha = \bar{X}^\alpha \tilde{T}, \qquad \tilde{\tilde{X}}^\alpha = \bar{\bar{X}}^\alpha + \bar{X}^\alpha \tilde{\tilde{T}}. \tag{4.11}$$

5. The quadrupole moments

The quadrupole moments of a comoving region V at constant time T are

$$D^{\alpha\beta}=\int_{V,T=\mathrm{const}}\rho(X,Y,Z,T)[3X^\alpha X^\beta-\delta^{\alpha\beta}X^\mu X^\mu]\,\mathrm{d}X\,\mathrm{d}Y\,\mathrm{d}Z$$

$$=\int_{V,T=\mathrm{const}}\rho(X(r,y,z,T),Y(r,y,z,T),Z(r,y,z,T),T)$$
$$\times[3X^\alpha(r,y,z,T)X^\beta(r,y,z,T)-\delta^{\alpha\beta}X^\mu X^\mu(r,y,z,T)]$$
$$\times{}^3J(r,y,z,T)\,\mathrm{d}r\,\mathrm{d}y\,\mathrm{d}z, \tag{5.1}$$

with

$${}^3J=\left|\frac{\partial(X,Y,Z)}{\partial(r,y,z)}\right|_{T=\mathrm{const}} \tag{5.2}$$

Let $\tilde{\eta}_{ik}$ denote the metric of the flat space–time in the coordinates r, y, z, T. The (four-dimensional) Jacobian for the transformation (4.10) is

$$J=\left|\frac{\partial(X,Y,Z,T)}{\partial(r,y,z,T)}\right|={}^3J, \tag{5.3}$$

and hence, given that

$$(-\tilde{\eta})^{1/2}\,\mathrm{d}r\,\mathrm{d}y\,\mathrm{d}z\,\mathrm{d}T=\mathrm{d}X\,\mathrm{d}Y\,\mathrm{d}Z\,\mathrm{d}T, \tag{5.4}$$

where $\tilde{\eta}$ is the determinant of the matrix $\tilde{\eta}_{ik}$, we have

$$(-\tilde{\eta})^{1/2}={}^3J. \tag{5.5}$$

To evaluate $\tilde{\eta}$, let us consider the transformation $r,y,z,t\rightarrow r,y,z,T$. We have from (4.8)

$$(-\tilde{\eta})^{1/2}=(-\eta)^{1/2}\left|\frac{\partial(r,y,z,t)}{\partial(r,y,z,T)}\right|=|\tilde{T}|(-\eta)^{1/2}. \tag{5.6}$$

η is, from (2.2) and (2.3),

$$\eta=-\frac{\phi^4}{P^2W^2}\left[\left(\frac{\phi}{P}\right)_1\right]^2, \tag{5.7}$$

where we have written ϕ instead of ϕ^0, and therefore

$$(-\tilde{\eta})^{1/2}=e\left(\frac{\tilde{T}\phi^2}{PW}\right)\left(\frac{\phi}{P}\right)_1,\qquad e=\pm1. \tag{5.8}$$

Finally, we see from (4.10) that the term in square brackets in the quadrupole formula (5.1) is of the form

$$3X^\alpha X^\beta-\delta^{\alpha\beta}X^\mu X^\mu=F^{\alpha\beta}(r,y,z)T^2+G^{\alpha\beta}(r,y,z)T+H^{\alpha\beta}(r,y,z). \tag{5.9}$$

Inserting (3.9), (5.8) and (5.9) in (5.1) and omitting the parameter ϵ we obtain

$$D^{\alpha\beta}=\frac{e}{8\pi}\int_V\left[\frac{(P\tilde{S}_1-3\tilde{S}P_1)\tilde{T}}{P^3W}\right](r,y,z)$$
$$\times[F^{\alpha\beta}(r,y,z)T^2+G^{\alpha\beta}(r,y,z)T+H^{\alpha\beta}(r,y,z)]\,\mathrm{d}r\,\mathrm{d}y\,\mathrm{d}z. \tag{5.10}$$

Since the description of V depends only on r, y, z and not on T, it is readily apparent that

$$\dddot{D}^{\alpha\beta}=0, \tag{5.11}$$

where the dot denotes differentiation with respect to T.

We conclude that according to Einstein's quadrupole formula (1.1) the matter in Szekeres' quasi-spherical space–times does not radiate. The result is independent of the shape of the moving mass.

6. Conclusion

This work shows that for quasi-spherical space–times the linear approximation agrees with the exact solution in predicting that a body of dust moving in a Szekeres régime does not radiate gravitational waves. The result of the linear approximation is independent of the shape of the body. In this respect it is more general than that of the exact solution of Bonnor which referred only to a spherical body.

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